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Quantum group related to the space of differential operators on the quantum hyperplane

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Abstract. Quantum group $\mathcal{A}_{qp}$ is constructed and investigated. The Hopf algebra structure of the differential calculus on the quantum Euclidean space is given. Elements of the theory of representation of $\mathcal{A}_{qp}$ are presented in comparison with the general method described by Fadeev, Reshetikhin and Takhtajan. $\mathcal{A}_{qp}$ at roots of unity is analysed.

1. Introduction

In recent years quantum groups have attracted the attention of a large group of mathematicians and physicists. From the mathematical point of view quantum groups are understood in two different ways: as (quasi-triangular) Hopf algebras [1, 4, 6] or as C^* algebras [16]. In the former the concept of quantum groups arises from the quantum method of solving the inverse problem [7] and from the effective methods for solving the quantum Yang–Baxter equation [5, 8]; in the latter the quantum group becomes an interesting example of operator algebra and non-commutative geometry [11].

On the other hand differential structures on quantum groups [15] and quantum spaces [14] are of great practical importance. They arise from the concept of non-commutative differential geometry [3]. The quantum hyperplane is the simplest example of a non-commutative space. Differential structures on the quantum hyperplane were classified in [2]. In this article we construct a non-commutative and non-co-commutative Hopf algebra (quantum group) $\mathcal{A}_{qp}$ which is an N-dimensional generalization of the Hopf algebra considered in [13]. The Hopf algebra $\mathcal{A}_{qp}$ is isomorphic to the algebra $\mathcal{O}$ of differential operators related to the calculus on the N-dimensional quantum hyperplane. In this way we prove that $\mathcal{O}$ is a Hopf algebra and answer (partially) the question asked by Manin in [10].

We also present a representation theory of $\mathcal{A}_{qp}$ and discuss the possibility of a construction of $\mathcal{A}_{qp}$ in the case when the deformation parameters are roots of unity.

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2. Construction of $\mathcal{A}_{qp}$

Let $\mathcal{A}_{qp} = \mathbb{C}[[X_1, \dots X_N, P_1, \dots P_N]]/I_{qp}$ (formal power series in $2N$ variables) $N \in \mathbb{N}$, where I_{qp} is a two-sided ideal in $\mathbb{C}[[X_1, \dots, X_N, P_1, \dots, P_N]]$ given by the following relations

$$X_i X_j = q_{ij} X_j X_i \qquad P_i P_j = P_j P_i \tag{1}$$

$$X_i P_j = q_{ij} P_j X_i \ (i \neq j) \qquad X_i P_i = p_i P_i X_i \tag{2}$$

where $p_i, q_{ij} \in \mathbb{C} - \{0\}$, $q_{ij} = q_{ji}^{-1}$, $i, j = 1, \dots, N$. $\mathcal{A}_{qp}$ is an associative algebra with unity I. For the time being we assume that all parameters are not roots of unity. Introducing C-linear maps $\Delta_{\mathrm{R}}, \Delta_{\mathrm{L}} : \mathcal{A}_{qp} \to \mathcal{A}_{qp} \otimes \mathcal{A}_{qp}$ and $\epsilon : \mathcal{A}_{qp} \to \mathbb{C}$ defined as

$$\begin{aligned} &\Delta_{\mathrm{R}}(X_i) = X_i \otimes I + P_i \otimes X_i \qquad \Delta_{\mathrm{L}}(X_i) = I \otimes X_i + X_i \otimes P_i \\ &\Delta_{\mathrm{R,L}}(P_i) = P_i \otimes P_i \qquad \Delta_{\mathrm{R,L}}(I) = I \otimes I \\ &\epsilon(X_i) = 0 \qquad \epsilon(P_i) = \epsilon(I) = 1 \end{aligned} \tag{3}$$

we equip $\mathcal{A}_{qp}$ with a bialgebraic structure (cf [1]) (i.e. $\Delta_{\mathrm{R,L}}, \epsilon$ are algebra homomorphisms). Furthermore, if we assume that all P_i are invertible and define C-linear maps $S_{\mathrm{L}}, S_{\mathrm{R}} : \mathcal{A}_{qp} \to \mathcal{A}_{qp}$ such that:

$$S_{\mathrm{R}}(P_i) = P_i^{-1} \qquad S_{\mathrm{R}}(X_i) = -P_i^{-1} X_i \qquad S_{\mathrm{R}}(I) = I \tag{4}$$

$$S_{\mathrm{L}}(P_i) = P_i^{-1} \qquad S_{\mathrm{L}}(X_i) = -X_i P_i^{-1} \qquad S_{\mathrm{L}}(I) = I \tag{5}$$

then we can make $(\mathcal{A}_{qp}, \Delta_{\mathrm{R}}, \epsilon, S_{\mathrm{R}})$, $(\mathcal{A}_{qp}, \Delta_{\mathrm{L}}, \epsilon, S_{\mathrm{L}})$ into Hopf algebras (quantum groups).

Let us denote the set of all bialgebras $(\mathcal{A}_{qp}, \Delta_{\mathrm{R}}, \epsilon)$ ($(\mathcal{A}_{qp}, \Delta_{\mathrm{L}}, \epsilon)$ respectively) by $\mathbf{Big}_{\mathrm{R}}$, ($\mathbf{Big}_{\mathrm{L}}$ respectively) and the set of all Hopf algebras $(\mathcal{A}_{qp}, \Delta_{\mathrm{R}}, \epsilon, S_{\mathrm{R}})$ ($(\mathcal{A}_{qp}, \Delta_{\mathrm{L}}, \epsilon, S_{\mathrm{L}})$ respectively) by $\mathbf{Hopf}_{\mathrm{R}}$ ($\mathbf{Hopf}_{\mathrm{L}}$ respectively). If $\mathcal{A}_{qp} \in \mathbf{Hopf}_{\mathrm{R}}$ is generated by the set $\{X_i, P_i\}$ then elements $Y_i = P_i^{-1} X_i$, $Q_i = P_i$ generate an algebra $\mathcal{A}_{q^{-1}p^{-1}} \in \mathbf{Hopf}_{\mathrm{L}}$. Hence the sets $\mathbf{Hopf}_{\mathrm{R}}$ and $\mathbf{Hopf}_{\mathrm{L}}$ are isomorphic. Hence we can restrict ourselves to the elements of $\mathbf{Big}_{\mathrm{R}}, \mathbf{Hopf}_{\mathrm{R}}$ only. To avoid unnecessary complications in the notation we omit the subscripts R.

Due to [6] (cf [12]) we say that $\mathcal{A}_{qp}^{\mathbb{C}}$ is a *complexification* of $\mathcal{A}_{qp}$ if $\mathcal{A}_{qp}^{\mathbb{C}}$ is a *-Hopf algebra (i.e. a Hopf algebra with involution). We also say that algebra $\mathcal{A}_{qp}^{\mathbf{R}}$ is a *realification* of $\mathcal{A}_{qp}$ if $\mathcal{A}_{qp}^{\mathbf{R}}$ is a *-Hopf algebra and $X_i^* = X_i$, $P_i^* = P_i$ for any $i = 1, \dots, N$. One can easily see that $\mathcal{A}_{qp} \in \mathbf{Hopf}$ admits realification if and only if $q_{ij}, p_i \in S^1$. If, moreover, N is even and $q_{i'j'} = q_{ji}^*$, $q_{i'j} = q_{j'i}^*$, $p_i p_{i'}^* = 1$, where $i' = N + 1 - i$, $j' = N + 1 - j$, $i, j = 1, \dots, N$ then $\mathcal{A}_{qp}$ can be complexified in such a way that $X_i^* = X_{i'}$, $P_i^* = P_{i'}$.

3. Differential operators on the quantum Euclidean space

As is well known [11] the N-dimensional quantum Euclidean space (or, precisely, the space of functions on the N-dimensional quantum hyperplane) is understood to be the algebra

$$\mathbb{C}_q^N = \mathbb{C}[[x^1, \dots, x^N]]/(x^i x^j - q_{ij} x^j x^i)$$

where $q_{ij} = q_{ji}^{-1} \in \mathbb{C} - \{0\}$, $i, j = 1, \ldots, N$. To introduce differential structure on the quantum hyperplane we have to define the C-linear, nilpotent operation d on $\mathbb{C}_q^N$, taking values in the $\mathbb{C}_q^N$-bimodule $\Lambda^1\mathbb{C}_q^N$, generated by the elements $\mathrm{d}x^i$, i.e. any element $\omega \in \Lambda^1\mathbb{C}_q^N$ is of the form $\sum_{k=1}^N \mathrm{d}x^k\omega_k$, where $\omega_k \in \mathbb{C}_q^N$. To make d into an exterior differential we have to require that d obeys the Leibniz rule. Notice that since $\Lambda^1\mathbb{C}_q^N$ is a $\mathbb{C}_q^N$-bimodule, every element $\omega \in \Lambda^1\mathbb{C}_q^N$ can also be represented as $\sum_{k=1}^N \omega_k' \mathrm{d}x^k$. In general $\omega_k' \neq \omega_k$.

Having defined exterior differentiation one can define derivatives (right and left) on $\mathbb{C}_q^N$, i.e. the linear operators $\mathrm{D}_k^\mathrm{R}, \mathrm{D}_k^\mathrm{L} : \mathbb{C}_q^N \to \mathbb{C}_q^N$ such that

$$\mathrm{d}f = \sum_{k=1}^N \mathrm{d}x^k \mathrm{D}_k^\mathrm{R} f \qquad \mathrm{d}f = \sum_{k-1}^N \mathrm{D}_k^\mathrm{L} f\, \mathrm{d}x^k$$

for any $f \in \mathbb{C}_q^N$. Differential calculus being non-commutative implies that left derivatives differ from the right ones. Moreover derivatives do not obey the Leibniz rule. Instead one finds that there exist operators $T_{ij}^R, T_{ij}^L : \mathbb{C}_q^N \to \mathbb{C}_q^N$, $i, j = 1, \ldots, N$ such that

$$\mathrm{D}_i^\mathrm{R}(fg) = \mathrm{D}_i^\mathrm{R} f g + \sum_{j=1}^N T_{ij}^\mathrm{R} f \mathrm{D}_j^\mathrm{R} g$$

$$\mathrm{D}_i^\mathrm{L}(fg) = f\mathrm{D}_i^\mathrm{L} g + \sum_{j=1}^N \mathrm{D}_j^\mathrm{L} f T_{ij}^\mathrm{L} g$$

$(f, g \in \mathbb{C}_q^N)$.

According to [2] there exists a differential calculus (family I) over $\mathbb{C}_q^N$ given by the relations:

$$x^i\,\mathrm{d}x^j = q_{ij}\,\mathrm{d}x^j x^i \;\; (i \neq j) \qquad x^i\,\mathrm{d}x^i = p_i\,\mathrm{d}x^i x^i \tag{6}$$

where $p_i \in \mathbb{C} - \{0\}$. Right derivatives take the form:

$$\mathrm{D}_i^\mathrm{R} f(x^1, \ldots, x^N) = \frac{1}{x^i} \frac{f(x^1, \ldots, x^{i-1}, p_i x^i, x^{i+1}, \ldots, x^N) - f(x^1, \ldots, x^N)}{p_i - 1}$$

where

$$f(x^1, \ldots, x^N) = \sum_{i_1, \ldots, i_N} f_{i_1 \ldots i_N} (x^1)^{i_1} \cdots (x^N)^{i_N}$$

$f_{i_1 \ldots i_N} \in \mathbb{C}$. The symbol $1/x^i$ means that one has to transport the variable x^i to the left and then decrease the power of x^i by one. It is not difficult to establish that $\mathrm{D}_i^\mathrm{R}\mathrm{D}_j^\mathrm{R} = q_{ij}\mathrm{D}_j^\mathrm{R}\mathrm{D}_i^\mathrm{R}$ and that matrix $T^\mathrm{R} = (T_{ij}^\mathrm{R})$ has the form $T^\mathrm{R} = \mathrm{diag}(T_1^\mathrm{R}, \ldots, T_N^\mathrm{R})$, where

$$T_i^\mathrm{R} f(x^1, \ldots, x^N) = f(q_{1i}x^1, q_{2i}x^2, \ldots, q_{i-1i}x^{i-1}, p_i x^i, q_{i+1i}x^{i+1}, \ldots, q_{Ni}x^N)$$

Putting $w = (x^1)^{n_1} \cdots (x^N)^{n_N}$ we get

$$D_i^R T_i^R(w) = p_i [n_i]_{p_i} (q_{1i} x^1)^{n_1} \cdots (q_{i-1i} x^{i-1})^{n_{i-1}} (p_i x^i)^{n_i - 1} \cdots (q_{Ni} x^N)^{n_N}$$

$$T_i^R D_i^R(w) = [n_i]_{p_i} (q_{1i} x^1)^{n_1} \cdots (q_{i-1i} x^{i-1})^{n_{i-1}} (p_i x^i)^{n_i - 1} \cdots (q_{Ni} x^N)^{n_N}$$

$$D_i^R T_j^R(w) = q_{ij} [n_i]_{p_i} (q_{1j} x^1)^{n_1} \cdots (q_{i-1j} x^{i-1})^{n_{i-1}} (q_{ij} x^i)^{n_i - 1} \cdots (q_{Nj} x^N)^{n_N}$$

$$T_j^R D_i^R(w) = [n_i]_{p_i} (q_{1j} x^1)^{n_1} \cdots (q_{i-1j} x^{i-1})^{n_{i-1}} (q_{ij} x^i)^{n_i - 1} \cdots (q_{Nj} x^N)^{n_N}$$

where $[n]_{p_i} = (p_i^n - 1)/(p_i - 1)$. This immediately allows us to establish commutation relations between Ds and Ts, namely:

$$T_i^R T_j^R = T_j^R T_i^R \qquad D_i^R T_i^R = p_i T_i^R D_i^R \qquad D_i^R T_j^R = q_{ij} T_j^R D_i^R \qquad (i \neq j). \quad (7)$$

Comparing (7) with (1) and (2) we see that the algebra $\mathcal{O}$ generated by the right derivatives D_k^R and matrix T^R is isomorphic to $\mathcal{A}_{qp} \in$ **Big**, hence it possesses a bialgebraic stucture. One can easily repeat this consideration, replacing right derivatives by the left ones and T^R by T^L. Moreover, since operators T_{ij}^R are invertible, the bialgebra $\mathcal{O}$ becomes Hopf algebra with the antipode defined by the relation (4).

4. Representations of $\mathcal{A}_{qp}$

Construction of the algebra $\mathcal{O}$ and its isomorphism to $\mathcal{A}_{qp}$ suggests an immediate representation of the latter in the space $\mathcal{F}_N$ of C-analytic functions of N-variables. If we denote by $\mathcal{L}(\mathcal{F}_N)$ the space of all linear operators in $\mathcal{F}_N$ then the representation $\pi : \mathcal{A}_{qp} \to \mathcal{L}(\mathcal{F}_N)$ in question is given by the relations

$$\pi(X_i) f(x^1, \ldots, x^N) = \frac{1}{x^i} \frac{f(x^1, \ldots, x^{i-1}, p_i x^i, x^{i+1}, \ldots, x^N) - f(x^1, \ldots, x^N)}{p_i - 1}$$

$$\pi(P_i) f(x^1, \ldots, x^N) = f(q_{1i} x^1, q_{2i} x^2, \ldots, q_{i-1i} x^{i-1}, p_i x^i, q_{i+1i} x^{i+1}, \ldots, q_{Ni} x^N)$$

$$\pi(P_i^{-1}) f(x^1, \ldots, x^N) = f(q_{i1} x^1, q_{i2} x^2, \ldots, q_{ii-1} x^{i-1}, p_i^{-1} x^i, q_{ii+1} x^{i+1}, \ldots, q_{iN} x^N).$$

In the theory of quantum groups (Hopf algebras) an important role is played by the notion of a co-module of the quantum group. If $\mathcal{A}$ is a Hopf algebra then the algebra V is called a *left $\mathcal{A}$-co-module* if there exists an algebra homomorphism $\delta : V \to \mathcal{A} \otimes V$ such that

$$(\mathrm{id}_{\mathcal{A}} \otimes \delta)\delta = (\Delta \otimes \mathrm{id}_V)\delta \qquad (\epsilon \otimes \mathrm{id})\delta = \mathrm{id}_V. \quad (8)$$

Usually such a co-module V is interpreted (cf [6]) as a quantum space, covariant with respect to the action of the quantum group $\mathcal{A}$. Any $\mathcal{A}_{qp} \in$ **Hopf** can be associated with a space $V = \mathbb{C}[[v_1, \ldots, v_{2N}]]/I_V$, where I_V is a two-sided ideal in the algebra $\mathbb{C}[[v_1, \ldots, v_{2N}]]$ generated by the elements

$$v_i v_{j+N} - v_{j+N} v_i \qquad v_{i+N} v_{j+N} - v_{j+N} v_{i+N} \qquad v_i v_j - q_{ij} v_j v_i \quad (9)$$

where $i, j = 1, \ldots, N$. The co-action δ is given by the relations

$$\delta(v_i) = P_i \otimes v_i + X_i \otimes v_{i+N} \qquad \delta(v_{i+N}) = I \otimes v_{i+N}. \tag{10}$$

Assume again that $\mathcal{A}$ is a Hopf algebra. According to [16] we say that $u = (u_{ij})_{i,j=1}^{K} \in M_K(\mathcal{A})$ (here $M_K(\mathcal{A})$ stands for the space of all $K \times K$ matrices with entries from $\mathcal{A}$) is a *K-dimensional co-representation of* $\mathcal{A}$ if

$$\Delta(u_{ij}) = \sum_{k=1}^{K} u_{ik} \otimes u_{kj} \qquad \epsilon(u_{ij}) = \delta_{ij}. \tag{11}$$

Two co-representations $u \in M_K(\mathcal{A})$, $w \in M_{\mathrm{L}}(\mathcal{A})$ are said to be *equivalent* if there exists an invertible element $A \in M_{L\times K}(\mathbb{C})$ such that $Au = wA$.

Applying these definitions to any $\mathcal{A}_{qp} \in$ **Hopf** one finds that the matrix $u \in M_{2N}(\mathcal{A}_{qp})$ with only non-zero entries:

$$u_{ii} = P_i \qquad u_{i+Ni+N} = I \qquad u_{ii+N} = X_i \tag{12}$$

$i = 1, \ldots, N$ is a co-representation of $\mathcal{A}_{qp}$. This co-representation is called a fundamental co-representation of $\mathcal{A}_{qp}$.

It is not difficult to obtain another $2N$-dimensional co-representation of $\mathcal{A}_{qp}$, namely $w \in M_{2N}(\mathcal{A}_{qp})$ with only non-vanishing elements $w_{ii} = I$, $w_{i+Ni+N} = P_i$, $w_{i+Ni} = X_i$. But co-representations w and u are equivalent in the sense described above, and the matrix $A \in M_{2N}(\mathbb{C})$ has only non-vanishing elements $a_{ii+N} = a_{i+Ni} = 1$, $i = 1, \ldots, N$.

Comparing the definition of the elements of **Big** with the procedure described by Faddeev *et al* [6] (via the R matrix and relations $Ru_1u_2 = u_2u_1R$) one can easily show that the R matrix giving the relations (1) and (2) does not exist for a general choice of parameters p_i, q_{ij}. This means that our Hopf algebra is not an algebra of functions on any quantum matrix algebra in the sense of [6]. Bialgebra $\mathcal{A}_{qp}$ can be obtained from the R-matrix if $p_i = 1$, $i = 1, \ldots, N$. Here, for u we take the fundamental representation of $\mathcal{A}_{qp}$ (12) and then $R \in M_{2N\otimes 2N}(\mathbb{C})$ consists of the following non-zero elements $(i, j = 1, \ldots, N)$:

$$R_{i+Nj+N,i+Nj+N} = R_{i+Nj,i+Nj} = R_{ji+N,ji+N} = 1 \qquad R_{ij,ij} = q_{ji}.$$

5. $\mathcal{A}_{qp}$ at roots of unity

In this section we assume that all parameters defining $\mathcal{A}_{qp}$ are roots of unity, i.e. that there exist integer numbers $n_{ij}, m_i > 1$ such that

$$(q_{ij})^{n_{ij}} = (p_i)^{m_i} = 1 \tag{13}$$

(we assume that n_{ij} and m_i are the smallest numbers obeying (13)). Condition 'q_{ij}, p_i are roots of unity' immediately implies that deformation parameters lie on the unit circle ($q_{ij}, p_i \in S^1$), hence it is possible to define the *-bialgebra $\mathcal{A}_{qp}^{\mathbf{R}}$ as a realification of $\mathcal{A}_{qp}$. If we now denote by M_i the least common multiple of the numbers n_{ij}, m_i, $j = 1, \ldots, N$ (i fixed) (i.e. $M_i = \mathrm{LCM}(m_i, n_{i1}, \ldots, n_{iN})$) then $P_i^{M_i}$, $i = 1, \ldots, N$ belongs to the centre of $\mathcal{A}_{qp}$. Since ϵ is an algebra homomorphism and $\epsilon(P_i) = 1$, $\epsilon(P_i^{M_i}) = 1$. This means that we can put $P_i^{M_i} = I$, therefore make any P_i invertible and define Hopf algebra structure on $\mathcal{A}_{qp}$. Using a similar argument one can establish that $X_i^{M_i}$ is forced to disapear in any irreducible representation of $\mathcal{A}_{qp}$. This condition seems to be strong and leads to the conditions which have to be put on p_i and q_{ij} in order to preserve the homomorphicity of Δ, i.e. $X_i^{M_i} = 0$ implies

$$\Delta(X_i)^{M_i} = 0. \tag{14}$$

In general we have

$$\Delta(X_i)^{M_i} = (X_i \otimes I + P_i \otimes X_i)^{M_i} = \sum_{k=0}^{M_i} c^i_{M_i k} {P_i}^k {X_i}^{M_i - k} \otimes {X_i}^k$$

where the coefficients $c^i_{nk} \in \mathbb{C}$, $n \in \mathbb{N}$, $k = 0, 1, \ldots, n$, $i = 1, 2, \ldots, N$ are defined inductively

$$c^i_{n+1k} = {p_i}^{n+1-k} c^i_{nk-1} + c^i_{nk} \qquad c^i_{n0} = c^i_{nn} = 1 \tag{15}$$

$k = 1, \ldots, n-1$, $n \in \mathbb{N}$, $i = 1, \ldots, N$. These equations have a well known solution:

$$c^i_{nk} = \frac{[n]_{p_i}!}{[n-k]_{p_i}! [k]_{p_i}!}$$

where $[n]_x! = [1]_x [2]_x \cdots [n]_x$ is a quantum factorial. Coefficients c^i_{nk} are then Gaussian polynomials or q-binomial coeffcients.

Condition (14) is then equivalent to the condition

$$c^i_{M_i k} = 0 \tag{16}$$

$k = 1, \ldots, M_i - 1$, $i = 1, \ldots, N$.

Let us consider polynomials $W_i \in \mathbb{C}[x]$, $i = 1, \ldots, N$

$$W_i(x) = \prod_{k=1}^{M_i} (x + {p_i}^k) = \sum_{k=0}^{M_i} d^i_{M_i k} x^{M_i - k}.$$

One can easily check that $f^i_{M_i k} = {p_i}^{-k(k+1)/2} d^i_{M_i k}$, $k = 0, 1, \ldots, M_i$ obey conditions (15) therefore $d^i_{M_i k} = {p_i}^{k(k+1)/2} c^i_{M_i k}$, hence condition (16) is equivalent to the following equation

$$W_i(x) = x^{M_i} + {p_i}^{(M_i+1)M_i/2} = \prod_{k=1}^{M_i} (x - \omega^i_k {p_i}^{(M_i+1)/2}) \tag{17}$$

where $(\omega^i_k)^{M_i} = -1$.

Equation (17) shows that condition (16) is equivalent to the existence of the permutations $\sigma_i : \{1, \ldots, M_i\} \to \{1, \ldots, M_i\}$ such that

$$\omega^i_{\sigma_i(k)} p_i^{(M_i+1)/2} = -p_i{}^k. \tag{18}$$

Comparison of the arguments of the left- and right-hand sides of (18) gives that (18) is equivalent to the following

$$\sigma_i(k) = [ka_i + (1 - a_i)(1 + M_i)/2] \operatorname{mod} M_i \tag{19}$$

where $a_i \in \{1, \ldots, M_i\}$ are such that $\arg(p_i) = (2\pi a_i)/M_i$, $i = 1, \ldots, N$. The functions σ_i are permutations if and only if

$$\text{LCD}(a_i, M_i) = 1 \tag{20}$$

(here the abbreviation LCD denotes the largest common divisor). This is the required condition which one has to put on the parameters p_i in order to preserve the homomorphicity of Δ. Notice that parameters q_{ij} can be completely freely chosen.

The last consideration can be summarized in the following proposition.

Proposition. Let $q_{ij}, p_i \in S^1$ be given by equation (13) and $M_i = \text{LCM}(m_i, n_{i1}, \ldots, n_{iN})$. Let $\arg(p_i) = 2\pi a_i/M_i$, where $a_i \in \{1, \ldots, M_i\}$. Then $\mathcal{A}_{qp}$ has a bialgebraic structure in the sense of (14) if and only if $\text{LCD}(M_i, a_i) = 1$, for any $i \in \{1, \ldots, N\}$.

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